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NORTH CAROLINA UNIV AT CHAPEL HILL CENTER FOR
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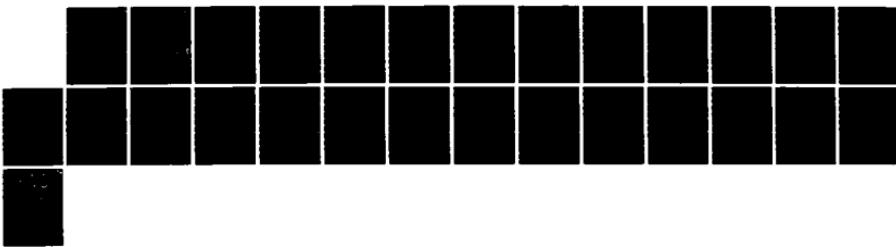
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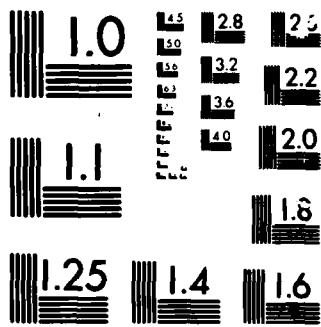
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Extension of Ito's Calculus via Malliavin Calculus

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by
A.S. Ustunel



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Introduction

This work is devoted to the development of the Itô's calculus for a class of functionals defined on the Wiener space which are more general objects than semimartingales. In doing this, we begin by an extension of the Itô formula for finite dimensional hypoelliptic Itô processes to the tempered distributions. In [14], S. Watanabe has defined the composition of a tempered distribution by a hypoelliptic Wiener functional with the help of the Malliavin Calculus. Here we go a little further and give an Itô formula by using the same method. Let us note that when the Itô process is the standard Wiener process, the Itô formula has already been extended to the tempered distributions with the use of the Hida calculus (cf. [2], [4]); we give here a different approach which works for more general processes than the standard Wiener process. In the extended Itô formula, the Lebesgue integral part can be interpreted as a Bochner integral in some Sobolev space on the Wiener space (cf. [3] and the notations); however the remaining part is not an ordinary stochastic integral, despite the fact that it corresponds to a functional in some Sobolev space on the Wiener space. This situation suggests an extension of the Itô stochastic integral to the objects which are not necessarily stochastic processes. This extension is realized in the third section, first by showing that the Itô integral is an isomorphism of the space of smooth processes onto the space of smooth Wiener functionals,

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then by extending it to continuous linear forms on the space of the smooth processes. Further applications we give also the energy inequalities corresponding to the Sobolev norms of the distribution-processes and their stochastic integrals. As an immediate consequence of this extension, we obtain an Itô Representation Theorem for the distributions on the Wiener space, which gives a Haussmann-Clark type formula (cf. [1], [9]) for second order Wiener distributions.

In the last section we discuss the integration by parts formula for Wiener distributions in which the directional derivatives are taken in the direction of Random Cameron-Martin vectors. In order to do this, we first prove that these random directional derivatives map the space of smooth Wiener functionals into itself. However, this is not sufficient to extend the integration by parts formula to the space of distributions on Wiener space. Curiously, for this extnesion we need a hypothesis of non-anticipation of the Lebesgue density of the random Cameron-Martin vector. With this supplementary hypothesis, we show that the integration by parts formula with random directional derivative can be extended to the space of distributions on Wiener space.

Let us note that some of the results of this work have been announced in the note [13].

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I. Notations and preliminaries

Let Ω be the Wiener space $C([0,1] \mathbb{R}^d)$ and H be the Cameron-Martin space, i.e., the set of absolutely continuous functions on $[0,1]$ with values in \mathbb{R}^d having a square integrable density and μ will denote the standard Wiener measure on Ω . $(F_t; t \in [0,1])$ denotes the canonical increasing family of σ -algebras on Ω , completed with respect to the Wiener measure μ . We provide H with its usual Hilbert space topology and identify its dual by itself. The infinitesimal generator of the Ω -valued Ornstein-Uhlenbeck process is denoted by A (cf. [5], [6]) and the operator $\sqrt{-A}$ will be represented by C . Γ is the bilinear form defined as $\Gamma(fg) = A(fg) - fAg - gAf$ for smooth functionals f and g defined on Ω . We define higher order gradients by the following recurrence relation (cf. [8]):

$$\Gamma_1(f, f) = \Gamma(f, f)$$

$$\Gamma_{kH}(f, f) = (1/2) A \Gamma_k(f, f) - \Gamma_k(f, Af) - k \Gamma_k(f, f).$$

In fact, $\Gamma_k(f, f)$ is nothing but the Hilbert-Schmidt norm of the k -linear operator $D^k f$ on the Cameron-Martin space H where $D^k f$ denotes the k -fold weak H -derivative of the smooth functional f (cf. [11]). $D_{p,k}$ denotes the Banach space which is the completion of the following normed space

$$\{f \in L^p(\mu): \sum_{i \leq k} \|A^i f\|_{L^p(\mu)} < +\infty\}$$

and D is the space of the test functionals:

$$D = \cap_{p,k} D_{p,k}$$

equipped with its projective topology. We recall that D is an algebra (cf. [7]). D' will denote the continuous dual of D and its elements are called the distributions on the Wiener space Ω .

Similarly, for the stochastic processes we define the following space:

$D_{p,k}$ is the completion of the normed space ($p > 1$, $k \in \mathbb{N}$)

$\{h \in L^0(dt \times d\mu) : h \text{ is adapted and}$

$$\sum_{i \leq k} E\left[\left(\int_0^1 |A^i h(s, \omega)|^2 ds\right)^{p/2}\right] = \|h\|_{D_{p,k}}^p < +\infty\}.$$

Afterwards we define D as

$$D = \bigcap_{p,k} D_{p,k}$$

equipped with the projective limit topology and D' denotes its continuous dual.

II. An extension of Itô's formula

In [14] S. Watanabe has defined the composition of an hypoelliptic \mathbb{R}^d -valued smooth Wiener functional with a tempered distribution on \mathbb{R}^d using Malliavin's calculus. Here we propose to go one step further extending his method. First we shall treat an almost (but not completely) trivial case: Suppose that $(x_t; t \in [0,1])$ is an Itô process with values in \mathbb{R} (we work with $d=1$), i.e., a semimartingale such that

$$dx_t(\omega) = b(t, \omega) dt + \sigma(t, \omega) dW(t, \omega)$$

and let f be a continuous, bounded function on \mathbb{R} with a bounded, continuous first derivative, but, suppose that its second derivative in the sense of the distributions is worse than a measure. Let (f_n) be an approximating sequence of f in the space of rapidly decreasing, C^∞ -functions $S(\mathbb{R})$ such

that (∂f_n) converges to ∂f uniformly and hence $(\partial^2 f_n)$ converges to $\partial^2 f$ in the space of the tempered distributions $S'(\mathbb{R})$. Using Itô's formula, we have

$$f_n(x(t)) - f_n(x(0)) - \int_0^t \partial f_n(x(s)) dx(s) = (1/2) \int_0^t a(s) \partial^2 f_n(x(s)) ds$$

where $a(s) = \delta^2(s)$. Furthermore, as in [14], we have

$$\Gamma(x(t), \partial f_n(x(t))) = \partial^2 f_n(x(t)) \Gamma(x(t), x(t)).$$

Supposing that $\Gamma^{-1}(x(t), x(t))$ exists, we obtain

$$\begin{aligned} \partial^2 f_n(x(t)) &= \Gamma^{-1}(x(t), x(t)) [A(x(t) \partial f_n(x(t))) - x(t) A \partial f_n(x(t)) \\ &\quad - \partial f_n(x(t)) A x(t)] \end{aligned}$$

Using this identity, for any test functional $\psi \in D$, we have

$$\begin{aligned} (II.1) \quad E[\psi \int_0^t a(s) \partial^2 f_n(x(s)) ds] &= \\ &= \int_0^t E[a(s) \Gamma^{-1}(x(s), x(s)) \{A(x(s) \partial f_n(x(s))) - \partial f_n(x(s)) A x(s) - \\ &\quad - x(s) A(\partial f_n(x(s)))\} \psi] ds. \end{aligned}$$

Using the fact that A is self-adjoint on $L^2(\mu)$ we see that (II.1) is equal to

$$\begin{aligned} \int_0^t & [\langle x(s) \partial f_n(x(s)), A(\Gamma^{-1}(x(s), x(s)) a(s) \psi) \rangle \\ & - \langle \partial f_n(x(s)) A x(s), a(s) \Gamma^{-1}(x(s), x(s)) \psi \rangle \\ & - \langle \partial f_n(x(s)), A(x(s) a(s) \Gamma^{-1}(x(s), x(s)) \psi) \rangle] ds. \end{aligned}$$

Consequently, if $(x(t); t \in [0,1])$ is "sufficiently smooth" and if $\Gamma^{-1}(x(t), x(t))$ is "sufficiently integrable" with respect to $dt \times du$, then the following limit

$$\lim_{n \rightarrow \infty} E \int_0^t \psi a(s) \partial^2 f_n(x(s)) ds = \lim_{n \rightarrow \infty} \langle \psi, \int_0^t a(s) \partial^2 f_n(x(s)) ds \rangle$$

exists for any test functional $\psi \in D$ and the corresponding continuous linear form on D can be written as a weak or Pettis integral:

$$\int_0^t a(s) \partial^2 f(x(s)) ds.$$

Therefore, modulo some regularity hypothesis on $(x(t); t \in [0,1])$, one has the following relation:

$$f(x(t)) - f(x(0)) - \int_0^t \partial f(x(s)) dx(s) = (1/2) \int_0^t a(s) \partial^2 f(x(s)) ds$$

where the right hand side is a continuous process but it is not an ordinary integral, it is a vector valued integral converging in D' .

In fact D' is too large and we can refine the calculations to find a smaller Sobolev space in which the integral converges. For this we have to study the three bilinear forms of the equality (II.1) as a linear form on D . For the first one, using the short-hand notations, we have

$$\begin{aligned} | \langle x \partial f(x), A(\psi \Gamma^{-1}(x,x)a) \rangle | &\leq \|x \partial f(x)\|_{D_{2,0}} \|A(\psi a \Gamma^{-1}(x,x))\|_{D_{2,0}} \leq \\ &\leq \|x \partial f(x)\|_{D_{2,0}} [\|\Gamma(\psi, a \Gamma^{-1}(x,x))\|_{D_{2,0}} + \|\psi A(a \Gamma^{-1}(x,x))\|_{D_{2,0}} \\ &\quad + \|a \Gamma^{-1}(x,x) A \psi\|_{D_{2,0}}] \leq \\ &\leq 3 \|x \partial f(x)\|_{D_{2,0}} \|\psi\|_{D_{4,1}} \left(\int_0^1 a(s) \Gamma^{-1}(x(s), x(s)) \|_{D_{4,1}}^4 ds \right)^{1/4} \end{aligned}$$

where we have used the inequalities of Hölder combined with the inequality

$$\|\Gamma^{1/2}(f, f)\|_{L^p(\mu)} \leq c_p \|Af\|_{L^p(\mu)} \quad (\text{cf. [6], [7], [8]}).$$

For the second and

third linear forms, using the same method, we obtain:

$$|\langle a \psi \Gamma^{-1}(x, x), \partial f(x) A x \rangle| \leq$$

$$c \|A x \partial f(x)\|_{D_{2,0}} \|\psi\|_{D_{4,1}} \left(\int_0^1 \|a(s) \Gamma^{-1}(x(s), x(s))\|_{D_{4,1}}^4 ds \right)^{1/4}$$

$$|\langle A(x \psi a \Gamma^{-1}(x, x)), \partial f(x) \rangle| \leq$$

$$d \|\partial f(x)\|_{D_{2,0}} \|\psi\|_{D_{4,1}} \left(\int_0^1 \|x(s) a(s) \Gamma^{-1}(x(s), x(s))\|_{D_{4,1}}^4 ds \right)^{1/4}$$

where c and d are some constants independent of ψ . We have proved:

Proposition II.1

Suppose that

$$\begin{aligned} \|\partial f(x) A x\|_{D_{2,0}} + \|x \partial f(x)\|_{D_{2,0}} + \int_0^1 \|a(s) \Gamma^{-1}(x(s), x(s))\|_{D_{4,1}}^4 ds + \\ + \int_0^1 \|x(s) a(s) \Gamma^{-1}(x(s), x(s))\|_{D_{4,1}}^4 ds < +\infty, \end{aligned}$$

then the integral

$$\int_0^1 a(s) \partial^2 f(x(s)) ds$$

is an element of $(D_{4,1})' = D_{4/3, -1}$ and it is strongly convergent, i.e., Bochner integrable in this Sobolev space.

Remark: We did not try to find the smallest Sobolev space but just gave an example to illustrate how to determine it.

What we have done above can be generalized to the space of the tempered distributions:

Theorem II.1

Suppose that x is a d -dimensional Itô process:

$$dx^i(t) = b_i(t, \omega)dt + \sigma_{ij}(t, \omega) dW^j(t, \omega), \quad i=1, \dots, d,$$

with

$$\begin{aligned} \sum_{i,j} E \int_0^1 \{ |A^k \sigma_{ij}(s)|^p + |A^k b_i(s)|^p + |\Gamma^{-1}(x^i(s), x^j(s))|^p \} ds + \\ + \sum_{i,j} E[|A^k x^i(0)|^p + |\Gamma^{-1}(x^i(0), x^j(0))|^p] \end{aligned}$$

for any $p \geq 1$, $k \in \mathbb{N}$, where $\Gamma^{-1}(x^i(t), x^j(t))$ is the element indexed with (i, j) of the inverse of the matrix $\{\Gamma(x^k(t), x^p(t)); k, l=1, \dots, d\}$.

Then, for any tempered distribution $T \in S'(\mathbb{R}^d)$, the mapping $t \mapsto T \circ x(t)$ is a well defined, $\sigma(D', D)$ -continuous mapping and it can be represented as the sum of two functionals:

$$T(x(t)) - T(x(0)) = J(t) + \sum_{i,j} \int_0^t [b_i(s) \partial_i T(x(s)) + (1/2) a_{ij}(s) \partial_{ij} T(x(s))] ds$$

where the integral is taken in D' as a Bochner integral, $J(t)$ is an element of D' such that, for any $\psi \in D$ which is F_s -measurable with $s < t$, one has

$$\langle J(t), \psi \rangle = \langle J(s), \psi \rangle .$$

Proof:

Before to proceed to the proof, let us note that, using Doob and Hölder inequalities, we have

$$\sum_{i,j \leq d} E \int_0^1 |A^k x^i(s)|^p ds ,$$

for any $k \in \mathbb{N}$ and $p \geq 1$.

For the proof let us remark that $\{\Gamma(x^i(t), x^j(t)); i, j \leq d, t \in [0, 1]\}$

is a matrix-valued Itô process hence it has continuous trajectories.

In fact, we have (cf. [7], [12]):

$$\begin{aligned}\Gamma(x^i(t), x^j(t)) &= (x^i(0), x^j(0)) + \sum_k \int_0^t [\Gamma(x^j(s), \sigma_{ik}(s) + \\ &\quad + \Gamma(x^i(s), \sigma_{jk}(s))] dW^k(s) + \\ &\quad + \int_0^t [\Gamma(x^i(s), b_j(s)) + \Gamma(x^j(s), b_i(s)) + a_{ij}(s) + \\ &\quad + \sum_k \Gamma(\sigma_{ik}(s), \sigma_{jk}(s))] ds.\end{aligned}$$

$\Gamma^{-1}(x, x)$ exists $dt \times d\mu$ -a.s., since $\Gamma(x(., \omega), x(., \omega))$ has almost surely continuous trajectories, a continuity argument shows that μ -almost surely for any t , $\Gamma(x(t), x(t))$ is invertible with continuous trajectories and using Cramer's rule, Itô's formula, Doob's and Hölder's inequalities ($\Gamma^{-1}(x, x)$ is also an Itô process) we see that, for any $t \in [0, 1]$, $\Gamma^{-1}(x^i(t), x^j(t))$ belongs to $L^p(\mu)$ for any $p \geq 1$. Using the same inequalities we see also that, for any $t \in [0, 1]$, $x^i(t)$ belongs to D for $i=1, \dots, d$. Consequently, the hypothesis of [14] to define $T \circ x(t)$ are satisfied. As in [14], since T is a tempered distribution, there exists some $\alpha > 0$ such that $T \in S_{-\alpha}$ where $S_{-\alpha}$ is the completion of $S(\mathbb{R}^d)$ with respect to the norm $\|f\|_{-\alpha} = \|(-\Delta + |x|^2)^{-\alpha} f\|_{L^2(dx)}$. Let $m > 0$ be such that

$$(-\Delta + |x|^2)^{-m} (S_{-\alpha}) \hookrightarrow C_b^2(\mathbb{R}^d)$$

where $C_b^2(\mathbb{R}^d)$ denotes the space of the continuous functions whose first two derivatives are continuous and bounded. Since $S(\mathbb{R}^d)$ is dense in $S_{-\alpha}$, there exists $(f_n)_{n \in \mathbb{N}} \subset S(\mathbb{R}^d)$ converging to T in $S_{-\alpha}$.

Consequently

$$\phi_n = (-\Delta + |x|^2)^{-m} f_n \xrightarrow{n \rightarrow \infty} (-\Delta + |x|^2)^{-m} \phi$$

in $C_b^2(\mathbb{R}^d)$. Let $K(s)$ be the random partial differential operator

$$K(s, \omega) = b_i(s, \omega) \partial_i + (1/2) a_{ij}(s, \omega) \partial_{ij}.$$

For any $\psi \in D$, we have

$$\begin{aligned} \int_0^1 \langle (K(s)(-\Delta + |x|^2)^m \phi_n)(x(s)), \psi \rangle_{L^2(\mu)} ds &= \\ &= \int_0^1 \langle \phi_n(x(s)), p(s, \psi) \rangle ds \end{aligned}$$

using the Malliavin Calculus (cf. [14]), where

$$p(s, \psi) = \sum_{i_1, \dots, i_m < +\infty} H_{i_1} (A(\dots (A(H_{i_{m-1}} A(H_{i_m}(s)\psi) \dots)$$

and $H_{i_k}(s)$ is a polynomial of $\Gamma^{-1}(x^i(s), x^j(s))$, $x^i(s)$, $Ax^i(s)$, $a_{ij}(s)$, $b_i(s)$, $\sigma_{ij}(s)$. By the hypothesis, we have

$$E \int_0^1 |A^k p(s, \psi)|^p ds < +\infty$$

For any p and k , hence the limit when n tends to infinity exists it is equal to

$$\int_0^1 \langle \phi(x(s)), p(s, \psi) \rangle_{L^2(\mu)} ds.$$

Because of the Bochner integrability of $s \mapsto p(s, \psi)$ in $L^2(\mu)$ and the boundedness of $s \mapsto \phi(x(s))$ in $L^\infty(\mu)$, we see that the continuous functionals on D

$$\psi \mapsto \int_0^t \langle \phi(x(s)), p(s, \psi) \rangle ds,$$

defined for each t , are continuous with respect to t for each fixed test functional $\psi \in D$. Hence the weak integral, so defined, is $\sigma(D', D)$ -continuous with respect to $t \in [0, 1]$. In fact, the integral exists in the sense of Bochner in some $D_{p, -k}$ by the hypothesis and the continuity with respect to t is valid in the strong topology of D' .

Let us now define $J(t)$ as

$$J(t) = T \circ x(t) - T \circ x(0) - \int_0^t K(s) T(x(s)) ds.$$

The mapping $t \rightarrow J(t)$ is also $\sigma(D', D)$ -continuous with values in D' .

To prove this assertion it is sufficient to prove the weak continuity of the mapping $t \rightarrow T \circ x(t)$: For any $\psi \in D$, we have

$$\langle T \circ x(t), \psi \rangle = \langle \phi(x(t)), q(t, \psi) \rangle$$

where

$$q(t, \psi) = \sum_{i_1, \dots, i_m < +\infty} G_{i_1} (A(\dots(A(G_{i_{m-1}} A(G_{i_m}(t)\psi)\dots)$$

and $G_{i_k}(t)$ is a polynomial of $\Gamma^{-1}(x^i(t), x^j(t))$, $x^i(t)$, $Ax^i(t)$, consequently, using Doob and Hölder inequalities and the hypothesis we see that the mapping $t \rightarrow \langle T \circ x(t), \psi \rangle$ is continuous for any $\psi \in D$.

Let us now look at the functional $J(t)$ more closely: By construction, for any test functional $\psi \in D$, we have

$$\begin{aligned} \langle J(t), \psi \rangle &= \lim_{n \rightarrow \infty} \langle \int_0^t \sigma_{ij}(s) \partial_i f_n(x(s)) dW^j(s), \psi \rangle \\ &= \lim_{n \rightarrow \infty} \langle J^n(t), \psi \rangle \end{aligned}$$

where, for any $n \in \mathbb{N}$, $(J^n(t); t \in [0, 1])$ is a square integrable martingale.

Consequently, if $\psi \in D$ is F_s -measurable with $s < t$, we have

$$\langle J(t), \psi \rangle = \lim_{n \rightarrow \infty} \langle J^n(t), \psi \rangle = \lim_{n \rightarrow \infty} \langle J^n(s), \psi \rangle = \langle J(s), \psi \rangle$$

and this completes the proof of the theorem.

///Q.E.D.

Remarks:

- i) The objects as $(J(t); t \in [0,1])$ will be called the pseudomartingales and their structure is studied in the next section.
- ii) What we have done can also be explained in the following way: Let $(h_n; n \in \mathbb{N})$ be the canonical basis of $S'(\mathbb{R}^d)$ consisting of the Hermite functions. If T is any tempered distribution, we have the following representation:

$$T = \sum_{i=1}^{\infty} \lambda_i h_i$$

where (λ_i) is a scalar sequence in some K the space depending how bad T is. Define $T \circ x(t)$ formally as

$$T \circ x(t) = \sum_i \lambda_i h_i \circ x(t),$$

develop $h_i \circ x(t)$ using Itô's formula for each i , commute the integrals of Itô's formula with the summation and then justify all this using Malliavin's calculus and the space of the Wiener distributions. In [10] this idea has been employed to obtain Itô's formula when $(x(t); t \in [0,1])$ is the standard Brownian motion and the justification is done with the help of Hida's calculus (cf. also [4]).

III. The extension of Itô's stochastic integral

In the preceding section we have seen that the weak limit of a sequence of stochastic integrals satisfies a restricted martingale property, where the restriction comes from the fact that the martingale property has to be tested on the smooth Wiener functionals. Consequently, it is natural to ask if we can represent the limiting distribution as a kind of stochastic integral as we represented the limit of Lebesgue integrals as a D' -valued Bochner integral. In order to answer this question, it is evident that we have to look for an extension of the stochastic integral of Itô to the elements of D' . This amounts up to characterize the stochastic integral as an isomorphism between the space of smooth Wiener functionals (or the test functionals) D and the space of the smooth processes D (cf. Section I for the notations). In the following we shall work with $d=1$. However, all the results are true for the higher dimensions under some obvious modifications.

Let us recall some well-known facts: The famous representation theorem of Itô says that any element F of $L^2(\mu)$ with zero expectation can be represented as the stochastic integral of a $dt \times d\mu$ -almost surely uniquely defined, adapted process in $L^2(dt \times d\mu)$, which will be denoted by $\partial_W F$ and let us note that we confound as usual the equivalence classes with their elements. In fact the mapping $F \rightarrow \partial_W F$ is nothing but the adjoint of the isometry of Itô defining the stochastic integral. We can now announce

Lemma III.1

Let ψ be a Wiener functional with zero mean. ψ belongs to D if and only if $\partial_W \psi$ belongs to D .

Proof:

If ψ belongs to D then, for any $k \in \mathbb{N}$, $u = ((1/2)I - A)^k \psi$ belongs also to D . Itô's representation theorem implies that the functional u can be represented as

$$u = \int_0^1 (\partial_W u)(s) dW(s) = ((1/2)I - A)^k \int_0^1 (\partial_W \psi)(s) dW(s)$$

where

$$E\left[\left(\int_0^1 |(\partial_W u)(s)|^2 ds\right)^{p/2}\right] < +\infty$$

for any $p \geq 1$ since u belongs to all of the H^p -spaces of the martingales (cf. [6]). We have

$$\begin{aligned} \psi &= ((1/2)I - A)^{-k} u = ((1/2)I - A)^{-k} \int_0^1 (\partial_W u)(s) dW(s) \\ &= \int_0^1 (I - A)^{-k} (\partial_W u)(s) dW(s) \end{aligned}$$

where the last equality follows from the commutation relations of A with the stochastic integrals (cf. [6], Theorem 2). Since $\partial_W \psi$ is uniquely defined, we have

$$\partial_W \psi = (I - A)^{-k} \partial_W u \quad dt \times d\mu \text{ -a.e.},$$

consequently

$$E\left[\left(\int_0^1 |(I - A)^k \partial_W \psi(s)|^2 ds\right)^{p/2}\right] = E\left[\left(\int_0^1 |\partial_W u(s)|^2 ds\right)^{p/2}\right] < +\infty,$$

by what we have explained above. Since p and k are arbitrary, this proves that $\partial_W \psi$ belongs to D .

If $\partial_W \psi$ belongs to D , then it is straightforward to see that ψ belongs to D with the help of the inequality of Burkholder-David-Gundy.

///Q.E.D.

Let us denote by D_0 the closed subspace of D defined by

$$D_0 = \{\psi - \langle \psi, 1 \rangle ; \psi \in D\}.$$

We have

Theorem III.1

The mapping $\partial_W: D_0 \rightarrow D$ is a topological isomorphism.

Proof:

According to the representation theorem of Itô, ∂_W is one-to-one. The fact that it is onto follows from Lemma III.1. Since D_0 and D are Fréchet spaces, an algebraic isomorphism is also topological.

///Q.E.D.

Corollary III.1

The mapping $J: D \rightarrow D_0$ defined by

$$J(h) = \int_0^1 h(s) dW(s)$$

is a topological isomorphism.

Proof:

J is the inverse of the mapping ∂_W .

///Q.E.D.

The following theorem gives the extension of J to the distribution processes:

Theorem III.2

The mapping $J: D \rightarrow D_0$ has a unique extension, denoted again by J , as an isomorphism from D' onto D'_0 where D'_0 is the closed subspace of D' defined by

$$D'_0 = \{F - \langle F, 1 \rangle ; F \in D'\}.$$

Proof:

Let h be any element of D' and $(h_n; n \in \mathbb{N}) \subset D$ a sequence converging to h in the weak topology $\sigma(D', D)$. If ψ is any element of D_0 , we have

$$\langle J(h_n), \psi \rangle = \langle \langle h_n, \partial_W \psi \rangle \rangle$$

where $\langle \langle \cdot, \cdot \rangle \rangle$ denotes the bilinear form corresponding to the dual pair (D, D') . Hence, by the hypothesis, the limit

$$\lim_{n \rightarrow \infty} \langle J(h_n), \psi \rangle$$

exists for any $\psi \in D_0$. We denote the corresponding linear form (on D_0) as $J(h)$. It remains to define $J(h)$ on the constant functionals as being zero. It is evident that such J is uniquely defined. If $J(h) = 0$, then, for any $\psi \in D_0$, we have

$$0 = \langle J(h), \psi \rangle = \langle \langle h, \partial_W \psi \rangle \rangle$$

and Theorem III.1 implies that $h = 0$, hence the mapping $J: D' \rightarrow D_0$ is one-to-one. To see that it is onto, let F be in D_0 , then, there exists a sequence $(F_n; n \in \mathbb{N})$ in D_0 converging to F in $\sigma(D', D)$. Consequently, for any $\psi \in D_0$, we have

$$\langle F, \psi \rangle = \lim_{n \rightarrow \infty} \langle F_n, \psi \rangle = \lim_{n \rightarrow \infty} \langle \langle \partial_W F_n, \partial_W \psi \rangle \rangle$$

so, by Theorem III.1, the weak limit of $(\partial_W F_n; n \in \mathbb{N})$ exists and it is uniquely defined. Let us denote it by $\partial_W F$. We have

$$\begin{aligned} \langle J(\partial_W F), \psi \rangle &= \langle \langle \partial_W F, \partial_W \psi \rangle \rangle = \lim_{n \rightarrow \infty} \langle \langle \partial_W F_n, \partial_W \psi \rangle \rangle \\ &= \lim_{n \rightarrow \infty} \langle F_n, \psi \rangle = \langle F, \psi \rangle \end{aligned}$$

for any $\psi \in D$ (we define $\partial_W \psi$ as $\partial_W(\psi - \langle \psi, 1 \rangle)$), therefore

$$F = (\partial_W F)$$

and this completes the proof.

///Q.E.D.

We have also proved

Corollary III.2 (Itô representation theorem)

Let F be any element of D' . There exists a uniquely defined element $\partial_W F$ of D' such that

$$F = \langle F, 1 \rangle + J(\partial_W F)$$

Proof:

It is sufficient to apply Theorem III.2 to $F - \langle F, 1 \rangle$.

///Q.E.D.

Remark:

One can see better that the operator J is the extension of Itô's integral by defining $D_{p,k}$ and $D_{p,k}$ by the following norms:

$$\|f\|_{D_{p,k}} = \| (I-A)^k f \|_{L^p(\mu)}$$

$$\|h\|_{D_{p,k}} = \left\| \left(\int_0^1 |((3/2)I - A)^k h(s)|^2 ds \right)^{1/2} \right\|_{L^p(\mu)},$$

then we have the following energy relations which are the consequences of the Burkholder-David-Gundy inequalities:

$$c(p) \|h\|_{D_{p,k}} \leq \|J(h)\|_{D_{p,k}} \leq c'(p) \|h\|_{D_{p,k}}$$

where $c(p)$ and $c'(p)$ are some universal constants depending only on p .

For $p = 2$ we have the equality:

$$\|h\|_{D_{2,k}} = \|J(h)\|_{D_{2,k}}$$

and for $p = 2, k = 0$, we obtain the classical integral of Itô.

Let us give a simple application of these results to Haussmann-Clark's formula: Suppose that F is any element of $\bigcup_{k=0}^{\infty} D_{2,k}'$. There

exists some $m > 0$ such that $(I-A)^{-m}F = G$ belongs to $D_{2,4}$ (for example).

Clark's formula says that (cf. [9]), we have

$$G = \langle F, 1 \rangle + \int_0^1 E \left[\frac{d}{d\tau} [DG](\tau) | F_\tau \right] dW(\tau)$$

where DG represents the weak H -derivative of G (cf. [11]). We can calculate then $\partial_W F$ of Corollary III.2, by applying $(I-A)^m$ to this relation:

$$\begin{aligned} F &= \langle F, 1 \rangle + (I-A)^m \int_0^1 E \left[\frac{d}{d\tau} [DG](\tau) | F_\tau \right] dW(\tau) \\ &= \langle F, 1 \rangle + \int_0^1 E \left[((3/2)I-A)^m \frac{d}{d\tau} [DG](\tau) | F_\tau \right] dW(\tau) \end{aligned}$$

and by the uniqueness result of Corollary III.2, we have

$$\partial_W F(\tau) = E \left[((3/2)I-A)^m \frac{d}{d\tau} [DG](\tau) | F_\tau \right]. \quad d\tau \times d\mu \text{ -a.s.}$$

Let us remark that the conditional expectation of an element of D' is well defined since A commutes with the conditional expectations (c.f. [7]).

IV. An extension of integration by parts formula

Suppose that ξ is an element of the Cameron-Martin space H ; then the directional derivative operator (in the direction of ξ) defined by $\nabla_\xi f = (Df, \xi)$, $f \in D$, where Df is the weak H -derivative of f and (\cdot, \cdot) denotes the scalar product in H , is a linear operator mapping D into itself (cf. [6]). Moreover, using, for instance, Girsanov-Cameron-Martin formula and a limiting procedure we see that (cf. [1], [5], [9])

$$\langle \nabla_\xi f, \psi \rangle + \langle f, \nabla_\xi \psi \rangle = \langle f, \psi \rangle \int_0^1 \dot{\xi}(s) \cdot dW(s)$$

for test functionals $f, \psi \in D$ where $\dot{\xi}$ is the Lebesgue density of ξ .

If F is any element of D' then we can extend the above relation by the usual limiting procedure to define $\nabla_\xi F$ as

$$\langle \nabla_\xi F, \psi \rangle = -\langle F, \nabla_\xi \psi \rangle + \langle F, \psi J(\xi) \rangle.$$

Note that, since D is an algebra and $J(\xi)$ belongs to D , $\nabla_\xi F$ is well defined. In this section we want to extend this relation to the case where the vector ξ is random. In order to do that we need the following

Lemma IV.1

Let $\xi : \Omega \rightarrow H$ be a random variable such that

$$\int_0^1 |C^k \xi(s)|^2 ds$$

belongs to all $L^p(\mu)$ for $p \geq 1$, for any $k \in \mathbb{N}$. We then have the following inequalities:

$$\begin{aligned} \left\| \|D^k \xi\|_{HS} \right\|_{L^p(\mu)} &\leq c(p) \left(E \left[\left(\int_0^1 |C^k \xi(s)|^2 ds \right)^{p/2} \right] \right)^{1/p} \leq \\ &\leq c'(p) \left[\left\| \|D^k \xi\|_{HS} \right\|_{L^p(\mu)} + \left\| \|\xi\|_H \right\|_{L^p(\mu)} \right] \end{aligned}$$

$p > 1$, $k \in \mathbb{N}$, where $c(p)$ and $c'(p)$ are some universal constants and $\|D^k \xi\|_{HS}$ denotes the Hilbert-Schmidt norm of the k -linear form $D^k \xi$ on H .

Proof:

This result is an extension of Theorem 2 of [8] to the vector case and the classical method of the proof is the use of the Rademacher series as it is indicated in [6] and [8]. First note that, using a complete, orthonormal basis (e_n) in H , we have

$$\|D^k \xi\|_{HS}^2 = \sum_{n=0}^{\infty} \Gamma_k((\xi, e_n), (\xi, e_n))$$

and obviously, it is sufficient to show that the $L^{p/2}(\mu)$ -norm of this

random variable can be controlled by

$$(E[(\sum_{n=0}^{\infty} (C^k(\xi, e_n))^2)^{p/2}])^{1/p}$$

Let $(r_n(t))$ be the Rademacher's random variables on $[0,1]$ and define

$$h(t) = \sum_n r_n(t) (\xi, e_n)$$

(at the beginning we can suppose that ξ depends only on finitely many coordinates and then we can pass to the limit). Let us also define the operator G as $f \rightarrow (\nabla_\alpha f; \alpha \in \mathbb{N}^k)$ for smooth functionals f , where

$\nabla_\alpha f = \nabla_{\alpha_1} \nabla_{\alpha_2} \dots \nabla_{\alpha_k} f$ if $\alpha = (\alpha_1, \dots, \alpha_k)$, $\nabla_{\alpha_i} = \nabla_{e_i}$. We have

$$\begin{aligned} \int_0^1 \|\Gamma_k^{1/2} (h(t), h(t))\|_{L^p(\mu)}^p dt &= \\ &= E[\int_0^1 dt \|\sum_n r_n(t) G(\xi, e_n)\|_{\ell_k^2}^p] \end{aligned}$$

where ℓ_k^2 is the Hilbert space $\ell^2 \times \dots \times \ell^2$ (k -times). Using Khintchine's inequality, we see that the right hand side of the above equality is equivalent to

$$\begin{aligned} E[(\sum_n \|G(\xi, e_n)\|_{\ell_k^2}^2)^{p/2}] &= E[(\sum_n \sum_{\alpha \in \mathbb{N}^k} (\nabla_\alpha(\xi, e_n))^2)^{p/2}] \\ &= E[(\sum_n \Gamma_k((\xi, e_n), (\xi, e_n)))^{p/2}]. \end{aligned}$$

On the other hand we know that (cf. [8], [3])

$$\begin{aligned} \int_0^1 \|\Gamma_k^{1/2}(h(t), h(t))\|_{L^p(\mu)}^p dt &\leq c(p) \int_0^1 \|C^k h(t)\|_{L^p(\mu)}^p dt \\ &= c(p) E[\int_0^1 |\sum_n r_n(t) C^k(\xi, e_n)|^p dt] \end{aligned}$$

and from the Khintchine's inequality the last term is equivalent to

$$c(p) E \left[\left(\sum_n (C^k(\xi, e_n))^2 \right)^{p/2} \right]$$

and this finishes the proof the first inequality, the second is proved with exactly the same method so we omit it.

///Q.E.D.

We can now announce

Theorem IV.1

Let ξ be as in Lemma IV.1, then ∇_ξ maps D into itself.

Proof:

It is sufficient to prove that $A^k \nabla_\xi \psi$ belongs to all $L^p(\mu)$ for any $\psi \in D$ and $k \in \mathbb{N}$. If we apply A^k to $\nabla_\xi \psi$, we have the sums of the following type

$$\sum_n \Gamma_k(\nabla_{e_n} \psi, (\xi, e_n)), \quad \sum_n \Gamma_{k-(i+j)}(A^i \nabla_{e_n} \psi, A^j(\xi, e_n)),$$
$$\sum_n A^i \nabla_{e_n} \psi \cdot A^j(\xi, e_n), \quad i + j \leq k.$$

With the help of Lemma IV.1, using Hölder's inequality the L^p -norms of these series can be controlled by the $D_{p,k}$ -norms of ψ and the seminorms of ξ defined as in the hypothesis of Lemma IV.1.

///Q.E.D.

Remark:

The above proof shows in fact that ∇_ξ is a continuous operator on D .

Let us now come back to the integration by parts formula: If $F \in D'$ then there exists $(F_n) \subset D$ converging to F in the weak topology $\sigma(D', D)$, and we have for a ξ as in the Lemma IV.1, provided that ξ is nonanticipative,

$$\langle \nabla_{\xi} F_n, \psi \rangle = - \langle F_n, \nabla_{\xi} \psi \rangle + \langle F_n, J(\dot{\xi}) \psi \rangle$$

where $J(\dot{\xi})$ is the stochastic integral of $\dot{\xi}$ and it belongs to D by

Corollary III.1. By Theorem IV.1, the limit as n tends to infinity exists:

$$\lim_{n \rightarrow \infty} \langle \nabla_{\xi} F_n, \psi \rangle = - \langle F, \nabla_{\xi} \psi \rangle + \langle F, J(\dot{\xi}) \psi \rangle$$

Consequently $\nabla_{\xi} F$ can be defined by

$$\nabla_{\xi} F = - \nabla_{\xi}^* F + J(\dot{\xi}) F.$$

Let us note that, in spite of the fact that $\nabla_{\xi}^* F$ is defined for anticipative $\dot{\xi}$, in order to define $\nabla_{\xi} F$ we have to suppose that $\dot{\xi}$ is nonanticipative.

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